A Version of Simpson's Rule for Multiple Integrals

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1 Introduction

Let $M(f)$ denote the Midpoint Rule and $T(f)$ the Trapezoidal Rule for estimating $\int_a^b f(x)dx$. Then $M(f)$ and $T(f)$ are each exact for $f(x) = 1$ and x. A more accurate rule can be obtained by taking $\lambda M(f) + (1-\lambda)T(f)$, where $\lambda = \frac{2}{3}$. Of course this rule is known as Simpson's Rule, and is exact for all polynomials of degree ≤ 3. The purpose of this paper is to extend this idea to multiple integrals over certain polygonal regions D_n in R^n. First, consider the case $n = 2$. So suppose that D is a polygonal region in R^2. The midpoint rule in one dimension is $M(f) = (b-a)f\left(\frac{a+b}{2}\right)$. A natural extension of this rule is

$$M(f) = A(D)f(P)$$

where $A(D) =$ area of D, and $P =$ centroid of D.

The trapezoidal rule in one dimension is $T(f) = (b-a)\left(\frac{f(a)+f(b)}{2}\right)$. A natural extension of that rule is

$$T(f) = A(D)\frac{1}{m}\sum_{k=1}^{m} f(P_k)$$

where the P_k are the vertices of D_n.

In general, let D_n be some polygonal region in R^n, let $P_0, ..., P_m$ denote the vertices of D_n, and let $P_{m+1} =$ center of mass of D_n. Define the linear functionals

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$M(f) = \mathrm{Vol}(D_n)\ f(P_{m+1}),\ T(f) = \mathrm{Vol}(D_n)(\frac{1}{m+1}\sum_{j=0}^{m} f(P_j))$, and for fixed $\lambda,\ 0 \leq \lambda \leq 1$,

$$L_\lambda = \lambda M(f) + (1-\lambda)T(f)$$

The idea is to choose λ so that L_λ is a good cubature rule(CR). Our objective here is not to provide **optimal** cubature rules, but to generalize the ideas from Simpson's Rule in one variable to several variables. In some cases we have reproduced known CRs using a different approach[1]. In other cases our CRs appear to be new. In addition, our approach suggest a general method for deriving CRs for polygonal domains.

For the n cube $[0,1] \times \cdots \times [0,1]$, $\lambda = \frac{2}{3}$, and the CR $L(f)$ is exact for all polynomials of degree ≤ 3(see CR3). This, of course, matches the degree of exactness of Simpson's Rule in one variable. For the n simplex, $\lambda = \frac{n+1}{n+2}$, and the CR $L(f)$ is exact for all polynomials of degree ≤ 2(see CR1).

For regular polygons in the plane with m sides, our approach fails for $m > 4$. For example, if $D =$ a regular hexagon, then **no** choice of λ makes L_λ exact for all polynomials of degree ≤ 2. Even for $m = 4$, if the polygon is **not regular**, then our approach fails as well. For example, let D be the trapezoid with vertices $\{(0,0),(1,0),(0,1),(1,2)\}$. Again, **no** choice of λ makes L_λ exact for all polynomials of degree ≤ 2. One can do better, however, by using points $Q_j \in \partial(D_n)$, **other than the vertices** of D_n, to generate $T(f)$. In some cases this leads to better formulas. For example, for the trapezoid D, this leads to a formula which is exact for polynomials of degree ≤ 2(see CR 5). Choosing points different from the vertices leads to a system of polynomial equations. We use Grobner Basis methods(see [2]), along with Maple, to solve such systems when possible. For the n Simplex T_n, we try using the center of mass of the faces of T_n to generate $T(f)$(see CR2). A similar idea works for the n cube.

Instead of using the weighted combination, $\lambda M(f) + (1-\lambda)T(f)$, another way to derive Simpson's Rule in one variable is to integrate the quadratic interpolant to f at $a, \frac{a+b}{2}$, and b. For the n Simplex T_n, our generalization L_λ can be obtained by integrating $p(\widehat{x})$ over T_n, where $p(\widehat{x})$ is the unique interpolant to $f(\widehat{x})$ at the vertices and center of mass of T_n. Indeed, $M(f) = \int_{T_n} T(\widehat{x})dV$, where $T(\widehat{x}) =$ tangent hyperplane to $f(\widehat{x})$ at the center of mass of T_n, and $T(f) = \int_{T_n} L(\widehat{x})dV$, where $L(\widehat{x}) =$ bilinear interpolant to $f(\widehat{x})$ at the vertices of T_n. This, of course, is the generalization from one variable,

[1]See [1] and [4] for a thorough treatment of CRs.

where $(b-a)f\left(\frac{a+b}{2}\right) = \int_a^b T(x)dx$, $T =$ tangent line to $f(x)$ at $\frac{a+b}{2}$, and the trapezoidal rule can be obtained by integrating the linear interpolant to f at the endpoints. For regions D_n in general, however, L_λ, T, and M do not always arise in this fashion. Indeed, the number of basis functions does not always match the number of nodes, and the interpolant may not be unique. This is what occurs for the n cube.

We also give an analgous formula, using four knots, for the unit disc(see CR 6).

Finally we note that for most of our CRs, all of the weights are positive, all of the knots lie inside the region D_n, and all but one of the knots lies on the boundary of the region.

2 n Simplex

Let $P_0, ..., P_n$ denote the $n+1$ vertices of the n simplex $T_n \subset R^n$. Hence $P_j = (0, 0, ..., 1, 0, 0)$ for $\dot{k} \geq 1$, and $P_0 = (0, 0, ..., 0), \widehat{x} = (x_1, ..., x_n), dV =$ standard Lebesgue measure on T_n. We list the following useful facts:

$$\text{Vol}(T_n) = \frac{1}{n!}, \; P_{n+1} = \text{Center of Mass of } T_n = (1/(n+1), ..., 1/(n+1))$$

$$\int_{T_n} x_k dV = \frac{1}{(n+1)!}, k = 1, ..., n$$

$$\int_{T_n} x_j x_k dV = \frac{1}{(n+2)!}, j \neq k$$

$$\int_{T_n} x_k^2 dV = \frac{2}{(n+2)!}, k = 1, ..., n$$

Define the following linear functionals:

$$M(f) = \text{Vol}(T_n)\ f(P_{n+1}) = \frac{1}{n!} f(1/(n+1), ..., 1/(n+1))$$

$$T(f) = \text{Vol}(T_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} f(P_j)\right) = \frac{1}{(n+1)!}\sum_{j=0}^{n} f(P_j)$$

$$I(f) = \int_{T_n} f(\widehat{x}) dV$$

Finally, for fixed $\lambda,\ 0 \leq \lambda \leq 1$, define

$$L_\lambda = \lambda M(f) + (1-\lambda)T(f)$$

First, if $f(\widehat{x}) = x_k$, then $I(f) = M(f) = T(f) = \frac{1}{(n+1)!}$ for any λ. Now let $f(\widehat{x}) = x_j x_k,\ j \neq k$. Then $I(f) = \frac{1}{(n+2)!}, M(f) = \frac{1}{(n+1)(n+1)!}, T(f) = 0$. Hence $L_\lambda(f) = I(f) \Rightarrow \lambda\left(\frac{1}{(n+1)(n+1)!}\right) = \frac{1}{(n+2)!} \Rightarrow \lambda = \frac{n+1}{n+2}$. Let $L = L_\lambda$,with this λ. Now, if $f(\widehat{x}) = x_k^2$, then $L(f) = \frac{n+1}{n+2}\frac{1}{n!}\frac{1}{(n+1)^2} + \frac{1}{n+2}\frac{1}{(n+1)!} = \frac{2}{(n+2)!} = I(f)$. We summarize

CR1: $L(f) = \dfrac{n+1}{(n+2)n!} f(P_{n+1}) + \dfrac{1}{(n+2)!}\displaystyle\sum_{j=0}^{n} f(P_j)$ is exact for all polynomials of degree ≤ 2.

L is *not* exact for all polynomials of degree ≤ 3. For example, if $f(x) = x_j x_k x_l, j \neq k, k \neq l, j \neq l$, then $I(f) = \frac{1}{(n+3)!}$, while $L(f) = \frac{1}{(n+1)(n+2)!} \neq I(f)$.

2.0.1 Connection with Interpolation

Another way to derive Simpson's Rule in one variable is by using quadratic interpolation at $a, \frac{a+b}{2}, b$. The cubature rule $L(f)$ above can also be obtained by integrating a certain second degree interpolant to f over T_n. Let

$$p(x_1, ..., x_n) = A_{n+1}x_1x_2 + \sum_{k=1}^{n} A_k x_k + A_0$$

We wish to choose the A_j so that

$$p(P_k) = f(P_k),\ k = 0, ..., n+1$$

It follows easily from (2.0.1) that $A_0 = f(P_0), A_k = f(P_k), k = 1, ..., n$, and

$A_{n+1} = f(P_{n+1}) - (n+1)\sum_{k=0}^{n} f(P_k)$. Hence $\int_{T_n} p(\widehat{x})dV = A_{n+2}\frac{1}{(n+2)!} + \frac{1}{(n+1)!}\sum_{k=1}^{n} A_k + \frac{1}{n!}A_0 = \frac{1}{(n+2)!}\left((n+1)^2 f(P_{n+1}) - (n+1)\sum_{k=0}^{n} f(P_k)\right)$

$+\frac{1}{(n+1)!}\left(\sum_{k=1}^{n} f(P_k) - nf(P_0)\right) + \frac{1}{n!}f(P_0) = \frac{(n+1)^2}{(n+2)!}f(P_{n+1}) + \frac{1}{(n+2)!}\sum_{k=0}^{n} f(P_k) = L(f)$.

Remark 1 *Note that one could use any $x_j x_k, j \neq k$ in constructing the interpolant p, and still obtain $\int_{T_n} p(\widehat{x})dV = L(f)$.*

Remark 2 *This, of course, generalizes the fact that Simpson's Rule in one variable can be obtained by integrating the quadratic interpolant to f at $a, b, \frac{a+b}{2}$. In more than one variable, however, our quadratic interpolant only uses one second degree basis function. Using the basis functions $\{1, x_1, ..., x_n, x_1x_2\}$, one obtains exactness for those functions as well as for $\frac{n^2+n-2}{2}$ additional quadratic terms.*

Remark 3 *One can also obtain the rules $M(f)$ and $T(f)$ using interpolation.*

$M(f) = \int_{T_n} T(\widehat{x})dV$, where T is the tangent hyperplane to f at P_{n+1}.

$T(f) = \int_{T_n} q(\widehat{x})dV$, where $q(x_1, ..., x_n) = \sum_{k=1}^{n} B_k x_k + B_0$, the B_k chosen so that

$q(P_k) = f(P_k),\ k = 0, ..., n.$

Again, this generalizes the fact that the midpoint rule in one variable can be obtained by integrating the tangent line at the midpoint, while the trapezoidal rule can be obtained by integrating the linear interpolant to f at the endpoints.

2.1 Other Boundary Points

It is interesting to examine what happens if we use other points on $\partial(S_n)$ to generate $T(f)$. We first examine $n = 2$ in some detail, and then generalize.

2.1.1 $n = 2$

Let $T_2 =$ the triangle in R^2 with vertices $\{(0,0), (1,0), (0,1)\}$. Use the points $(a, 0), (0, b), (c, 1-c), 0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1$ from each side of T_2 to define

$$T(f) = \text{Area}(T_2)\frac{1}{3}(f(a,0)+f(0,b)+f(c,1-c)) = \frac{1}{6}(f(a,0)+f(0,b)+f(c,1-c))$$

As earlier(with $(1/3, 1/3) =$ Center of mass of T),

$$M(f) = \text{Area}(T_2)f(1/3, 1/3) = \frac{1}{2}f(1/3, 1/3)$$

and, for fixed $\lambda,\ 0 \le \lambda \le 1$,

$$L_\lambda = \lambda M(f) + (1-\lambda)T(f)$$

- $L_\lambda(x) = \frac{1}{6}\lambda + \frac{1}{6}(1-\lambda)(a+c)$
- $L_\lambda(y) = \frac{1}{6}\lambda + \frac{1}{6}(1-\lambda)(b+1-c)$
- $L_\lambda(x^2) = \frac{1}{18}\lambda + \frac{1}{6}(1-\lambda)(a^2+c^2)$
- $L_\lambda(y^2) = \frac{1}{18}\lambda + \frac{1}{6}(1-\lambda)\left(b^2+(1-c)^2\right)$
- $L_\lambda(xy) = \frac{1}{18}\lambda + \frac{1}{6}(1-\lambda)c(1-c)$

We want to choose $a, b, c,$ and λ so that L_λ is exact for the functions x, y, x^2, y^2,and xy. Setting $L_\lambda(f) = I(f)$ yields the system of equations
$\frac{1}{6}\lambda + \frac{1}{6}(1-\lambda)(a+c) = \frac{1}{6}$, $\frac{1}{6}\lambda + \frac{1}{6}(1-\lambda)(b+1-c) = \frac{1}{6}$,
$\frac{1}{18}\lambda + \frac{1}{6}(1-\lambda)(a^2+c^2) = \frac{1}{12}$, $\frac{1}{18}\lambda + \frac{1}{6}(1-\lambda)\left(b^2+(1-c)^2\right) = \frac{1}{12}$,
$\frac{1}{18}\lambda + \frac{1}{6}(1-\lambda)c(1-c) = \frac{1}{24}$
We found the following Grobner basis using Maple:

$$\{-12c^2 + 12\lambda c^2 + 4\lambda - 3 + 12c - 12\lambda c, -1 + a + c, b - c\}$$

Setting each polynomial from the Grobner basis to 0 yields precisely the same solutions as the original system.

Hence $b = c$, $a = 1 - c$, and $\lambda = \frac{12c^2+3-12c}{-12c^2+4-12c}$, $1 - \lambda = \frac{1}{4}\frac{24c^2-1}{3c^2-1+3c}$

This gives $L_\lambda(f) = \lambda\frac{1}{2}f(1/3, 1/3) + (1-\lambda)\frac{1}{6}(f(1-c, 0) + f(0, c) + f(c, 1-c))$

Now we choose c so that L_λ is exact for quadratics. Since $L_\lambda(x^2) = \frac{1}{12}\frac{9c^2-1+3c-24c^3+24c^4}{3c^2-1+3c}$ and $\int_{T_2} x^2 dA = \frac{1}{12}$, $L_\lambda(x^2) = \frac{1}{12} \Rightarrow c = 0$ or $c = \frac{1}{2}$. $c = 0$ uses the vertices of T_2 to generate $T(f)$. Hence we use $c = \frac{1}{2} \Rightarrow \lambda = 0$. With this choice of c and λ, it follows that $L_\lambda(y^2) = \int_{T_2} y^2 dA$ and $L_\lambda(xy) = \int_{T_2} xy dA$.

We summarize

$$L(f) = \frac{1}{6}(f(\frac{1}{2}, 0) + f(0, \frac{1}{2}) + f(\frac{1}{2}, \frac{1}{2})) \text{ is exact for all polynomials of degree} \leq 2.$$

Remark 4 *The CR above is given in the well known book of Stroud(see [4]), as one of a group of formulas for the triangle* T_2*.*

Since $L(x^3) = \frac{1}{24} \neq \int_{T_2} x^3 dA = \frac{1}{20}$, L is not exact for all cubics.

Connection with Interpolation It is not hard to show that there is a unique interpolant, $p(x,y) = A_1xy + A_2x + A_3y + A_4$, to any given $f(x,y)$ at $(1/2,0),(0,1/2),(1/2,1/2),(1/3,1/3)$. It also follows easily that $\int_{T_2} p(x,y)dA = \frac{1}{6}(f(\frac{1}{2},0) + f(0,\frac{1}{2}) + f(\frac{1}{2},\frac{1}{2}))$, which is $L(f)$ given above.

2.1.2 General n

For T_2 above we used the midpoints of the edges for $T(f)$, so it is natural to try using the **center of mass of the faces** of T_n, given by $Q_k = (\frac{1}{n},\frac{1}{n},...,0,...,\frac{1}{n})$(all coordinates $\frac{1}{n}$ except the kth coordinate $= 0$), $k = 1,...,n$, $Q_{n+1} = (\frac{1}{n},\frac{1}{n},...,\frac{1}{n})$. This yields

$$T(f) = \frac{1}{(n+1)!}\sum_{k=1}^{n+1} f(Q_k)$$

As earlier,

$$M(f) = \frac{1}{n!} f(1/(n+1),...,1/(n+1))$$

$$L_\lambda = \lambda M(f) + (1-\lambda)T(f)$$

A simple computation shows that L_λ is exact for x_k for any λ,and is exact for $x_jx_k, j \neq k$, if $(\frac{1}{n+1})\lambda + (\frac{n-1}{n^2})(1-\lambda) = \frac{1}{n+2}$. This gives $\lambda = -(n-2)\frac{n+1}{n+2}$. With this λ, we have

$$\textbf{CR2}\text{: } L(f) = -(n-2)\frac{n+1}{n+2}\frac{1}{n!}f(1/(n+1),...,1/(n+1)) + \frac{n^2}{(n+2)!}\sum_{k=1}^{n+1} f(Q_k)$$

is exact for all polynomials of degree ≤ 2. Note that one of the weights is < 0 if $n \geq 3$.

Remark 5 *It would be nice to choose points on the faces which yield $\lambda \geq 0$, and in particular $\lambda = 0$. We tried this for $n = 3$, using points on each face of the simplex T_3. Letting $Q_1 = (a_1,a_2,0), Q_2 = (a_3,0,a_4), Q_3 = (0,a_5,a_6), Q_4 = (a_7,a_8,1-a_7-a_8)$, with*

$$a_j \geq 0\ \forall j,\ a_j + a_{j+1} \leq 1, j = 1,3,5,7$$

$$T(f) = \text{area}(T_3)\frac{1}{4}\sum_{j=1}^{4} f(Q_j) = \frac{1}{24}\sum_{j=1}^{4} f(Q_j)$$

$$M(f) = \frac{1}{6}f(1/4, 1/4, 1/4)$$

$$L_\lambda = \lambda M(f) + (1-\lambda)T(f)$$

The equations giving exactness for $x, y, z, x^2, y^2, z^2, xy, xz, yz$, after some simplification, look like $a_1+a_3+a_7 = 1, a_2+a_5+a_8 = 1, a_4+a_6-a_7-a_8 = 0$,
$\frac{1}{96}\ \lambda + (1-\ \lambda)\frac{1}{24}(a_1a_2 + a_7a_8) = \frac{1}{120}$,
$\frac{1}{96}\ \lambda + (1-\lambda)\frac{1}{24}(a_3a_4 + a_7(1-a_7-a_8)) = \frac{1}{120}$,
$\frac{1}{96}\ \lambda + (1-\ \lambda)\frac{1}{24}(a_5a_6 + a_8(1-a_7-a_8)) = \frac{1}{120}$,
$\frac{1}{96}\ \lambda + (1-\ \lambda)\frac{1}{24}(a_1^2 + a_3^2 + a_7^2) = \frac{1}{60}$,
$\frac{1}{96}\ \lambda + (1-\ \lambda)\frac{1}{24}(a_2^2 + a_5^2 + a_8^2) = \frac{1}{60}$,
$\frac{1}{96}\ \lambda + (1-\ \lambda)\frac{1}{24}(a_4^2 + a_6^2 + (1-a_7-a_8)^2) = \frac{1}{60}$
One can solve the first two equations for a_7 and a_8, and then substitute into the rest of the equations, yielding seven polynomial equations in seven unknowns. Of course two solutions to these equations are

- $a_1 = a_2 = a_4 = a_6 = a_7 = a_8 = 0,\ a_3 = a_4 = 1,\ \lambda = \frac{4}{5}$, which uses the vertices of T_3 for Q_j , and
- $a_j = \frac{1}{3}$ for all j, , $\lambda = -\frac{4}{5}$, which uses the center of mass of each face of T_3 for Q_j.

We applied Grobner Basis methods to the above equations giving exactness for $x, y, z, x^2, y^2, z^2, xy, xz, yz$, along with $\lambda = 0$. We were then able to show that there is no solution. That proves that one cannot make $\lambda = 0$. We were not able to find any positive values for λ, though it is possible that some exist.

3 Unit n Cube

Let $C_n = [0, 1]^n \subset R^n$, and let $P_0, ..., P_{m-1}$ denote the m vertices of $C_n, m = 2^n$. Assume that $P_0 = (0, ..., 0)$ and $P_{m-1} = (1, ..., 1)$. The other vertices have at least one coordinate which equals 0, and at least one coordinate which equals 1. Let $\widehat{x} = (x_1, ..., x_n), dV$ = standard Lebesgue measure on C_n. We list the following useful facts:

$$\mathrm{vol}(C_n) = 1, \text{ Center of Mass} = P_m = (1/2, \ldots, 1/2)$$

For any non–negative $r_1, ..., r_n$,

$$\int_{C_n} x_{k_1}^{r_1} x_{k_2}^{r_2} \cdots x_{k_j}^{r_j} dV = \frac{1}{r_1+1} \frac{1}{r_2+1} \cdots \frac{1}{r_j+1}$$

Define the following linear functionals:

$$M(f) = \mathrm{Vol}(C_n)\ f(P_m) = f(1/2, ..., 1/2)$$

$$T(f) = \mathrm{Vol}(C_n)\left(\frac{1}{m} \sum_{j=0}^{m-1} f(P_j)\right) = \frac{1}{m} \sum_{j=0}^{m-1} f(P_j)$$

$$I(f) = \int_{C_n} f(\widehat{x}) dV$$

Finally, for fixed $\lambda,\ 0 \le \lambda \le 1$,

$$L_\lambda = \lambda M(f) + (1-\lambda) T(f)$$

First, let $f(x) = x_k$. Then $M(f) = \frac{1}{2}$, and it is not hard to show that $T(f) = \frac{1}{m} 2^{n-1} = \frac{1}{2}$. Hence $M(f) = T(f) = I(f) \Rightarrow L_\lambda(f) = I(f)$ for any λ. Now let $f(x) = x_j x_k, j \ne k$. Then $M(f) = \frac{1}{4}$. since there are 2^{n-2} ways to get a 1 in both the jth and kth coordinates of a vertex of D, $T(f) = \frac{1}{m} 2^{n-2} = \frac{1}{4}$. Hence, again, $M(f) = T(f) = I(f)$ for any $\lambda \Rightarrow L_\lambda(f) = I(f)$. Also, if $f(x) = x_k^2$, then $M(f) = \frac{1}{4}$ and $T(f) = \frac{1}{2}$. Hence $L_\lambda(f) = I(f) = \frac{1}{3} \Rightarrow \lambda = \frac{2}{3}$. Now let $L = L_{2/3}$. Finally we consider third degree terms. First, if $f(x) = x_j x_k x_l, j \ne k, k \ne l, j \ne l$, then $T(f) = M(f) = I(f) = \frac{1}{8} \Rightarrow L(f) = I(f)$ for any λ. Second, if $f(x) = x_j^2 x_k, j \ne k$, then $T(f) = \frac{1}{4}, M(f) = \frac{1}{8}$, and $I(f) = \frac{1}{6}$, and hence $\frac{2}{3} M(f) + \frac{1}{3} T(f) = I(f)$. Finally, if $f(x) = x_k^3$, then
$T(f) = \frac{1}{2}, M(f) = \frac{1}{8}$, and $I(f) = \frac{1}{4}$, and again $\frac{2}{3} M(f) + \frac{1}{3} T(f) = I(f)$. We summarize

CR3: $L(f) = \dfrac{2}{3} f(1/2, ..., 1/2) + \dfrac{1}{3} \dfrac{1}{2^n} \displaystyle\sum_{j=0}^{2^n-1} f(P_j)$ is exact for all polynomials of degree ≤ 3.

Remark 6 *If $f(x) = x_k^4$, then $T(f) = \frac{1}{2}, M(f) = \frac{1}{16}$, and $I(f) = \frac{1}{5} \Rightarrow \frac{2}{3} M(f) + \frac{1}{3} T(f) \ne I(f)$.*

Hence L is *not exact* in general for polynomials of degree ≤ 4.

3.1 Other Points on Boundary of n Cube

3.1.1 $n = 2$

Let $D =$ unit square. The formulas above use the vertices of D to generate $T(f)$. Here we use the points $(a,0),(0,b),(c,1),(1,d)$ on $\partial(D)$, $0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1, 0 \le d \le 1$, to define

$$T(f) = \frac{1}{4}(f(a,0) + f(0,b) + f(c,1) + f(1,d))$$

As earlier,

$$M(f) = f(1/2, 1/2)$$

For $0 \le \lambda \le 1$,

$$L_\lambda = \lambda M(f) + (1-\lambda)T(f)$$

The idea is to choose a,b,c,d and λ so that L_λ is exact for all polynomials of degree ≤ 3.

- $L_\lambda(x) = \frac{1}{2}\lambda + \frac{1}{4}(1-\lambda)(a+c+1)$
- $L_\lambda(y) = \frac{1}{2}\lambda + \frac{1}{4}(1-\lambda)(b+1+d)$
- $L_\lambda(x^2) = \frac{1}{4}\lambda + \frac{1}{4}(1-\lambda)(a^2+c^2+1)$
- $L_\lambda(y^2) = \frac{1}{4}\lambda + \frac{1}{4}(1-\lambda)(b^2+1+d^2)$
- $L_\lambda(xy) = \frac{1}{4}\lambda + \frac{1}{4}(1-\lambda)(c+d)$
- $L_\lambda(x^3) = \frac{1}{8}\lambda + \frac{1}{4}(1-\lambda)(a^3+c^3+1)$
- $L_\lambda(y^3) = \frac{1}{8}\lambda + \frac{1}{4}(1-\lambda)(b^3+1+d^3)$
- $L_\lambda(x^2y) = \frac{1}{8}\lambda + \frac{1}{4}(1-\lambda)(c^2+d)$
- $L_\lambda(xy^2) = \frac{1}{8}\lambda + \frac{1}{4}(1-\lambda)(c+d^2)$

Letting $I(f) = \int_0^1 \int_0^1 f(x,y)dydx$, we have $I(x) = I(y) = \frac{1}{2}$, $I(x^2) = I(y^2) = \frac{1}{3}$, $I(xy) = I(x^3) = I(y^3) = \frac{1}{4}$, $I(x^2y) = I(xy^2) = \frac{1}{6}$, $I(x^2y^2) = \frac{1}{9}$, $I(x^3y) = I(xy^3) = \frac{1}{8}$, $I(x^4) = I(y^4) = \frac{1}{5}$. Setting $L_\lambda(f) = I(f)$ for the nine monomials above yields a system of nine polynomial equations in nine unknowns. Maple gives the Grobner Basis

$$\{3\lambda + 6d - 2 - 6\lambda d - 6d^2 + 6\lambda d^2, a - d, -1 + b + d, -1 + c + d\}$$

$a = d$, $b = 1 - d$, $c = 1 - d$, $\lambda = \frac{6d^2-6d+2}{6d^2-6d+3}$. We are free to choose d to force exactness for additional monomials. However, it is not possible to get exactness for all fourth degree polynomials. Say we want $L_\lambda(x^3y) = I(x^3y) = \frac{1}{8} \Rightarrow -\frac{1}{24}\frac{-9d^2+7d-3+2d^3}{2d^2-2d+1} = \frac{1}{8} \Rightarrow d = 0, d = 1, d = \frac{1}{2}$. $d = 0$ or $d = 1$ uses the vertices of D–the case $n = 2$ of the general method discussed earlier for the n cube. For $d = \frac{1}{2}$ we have $\lambda = \frac{1}{3}$. with this choice, L_λ is also exact for $f(x,y) = xy^3$. $L_\lambda(x^4) = \frac{5}{24} \neq I(x^4) = \frac{1}{5}$, so we do not get exactness for all polynomials of degree ≤ 4. We summarize

CR4: $L(f) = \dfrac{1}{3}f(1/2,1/2) + \dfrac{1}{6}(f(1/2,0) + f(0,1/2) + f(1/2,1) + f(1,1/2))$

is exact for all polynomials of degree ≤ 3. Note that $L(f)$ is also exact for x^3y and xy^3.

Remark 7 *It is not possible to choose d so that $\lambda = 0$, since $6d^2 - 6d + 2$ has no real roots.*

Remark 8 *For the n cube in general, $n \geq 3$, the natural extension would be to use the center of mass of the faces of C_n to generate $T(f)$. We leave the details to the reader.*

Remark 9 *Since $C_n = [0,1] \times \cdots \times [0,1]$, one could also use Simpson's Rule as a* ***product rule****. However, this does* ***not*** *yield any of the rules obtained in this section.*

3.2 Connection with Interpolation

Let

$$p(x,y) = A_1x^2 + A_2y^2 + A_3x + A_4y + A_5$$

Given $0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1, 0 \le d \le 1$, and $f(x,y)$, let
$P_1 = (a,0), P_2 = (0,b), P_3 = (c,1), P_4 = (1,d)$. We wish to choose the A_j so that

$$p(P_k) = f(P_k),\ k = 1, ..., 5$$

where $P_5 = (1/2, 1/2)$. The corresponding coefficient matrix is $B = \begin{pmatrix} a^2 & 0 & a & 0 & 1 \\ 0 & b^2 & 0 & b & 1 \\ c^2 & 1 & c & 1 & 1 \\ 1 & d^2 & 1 & d & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$.

If $a = 1/2$, $b = 1/2$, $c = 1/2$, $d = 1/2$, then $\det(B) = \frac{1}{16} \neq 0$. In this case there is a unique interpolant, and $B = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & 1 \\ \frac{1}{4} & 1 & \frac{1}{2} & 1 & 1 \\ 1 & \frac{1}{4} & 1 & \frac{1}{2} & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$. Then

$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix} = B^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = \begin{pmatrix} 2b_2 + 2b_4 - 4b_5 \\ 2b_1 + 2b_3 - 4b_5 \\ -3b_2 - b_4 + 4b_5 \\ -3b_1 - b_3 + 4b_5 \\ b_1 + b_2 - b_5 \end{pmatrix}$, where $b_1 = f(a,0), b_2 = f(0,b), b_3 = f(c,1), b_4 = f(1,d), b_5 = f(1/2,1/2)$. $\int_0^1 \int_0^1 (A_1x^2 + A_2y^2 + A_3x + A_4y + A_5)dxdy = \frac{1}{3}(A_1 + A_2) + \frac{1}{2}(A_3 + A_4) + A_5 =$

$\frac{1}{3}(2b_2 + 2b_4 - 4b_5 + 2b_1 + 2b_3 - 4b_5) + \frac{1}{2}(-3b_2 - b_4 + 4b_5 - 3b_1 - b_3 + 4b_5) + b_1 + b_2 - b_5 =$

$\frac{1}{6}b_2 + \frac{1}{6}b_4 + \frac{1}{3}b_5 + \frac{1}{6}b_1 + \frac{1}{6}b_3 = \frac{1}{3}f(1/2,1/2) + \frac{1}{6}(f(1/2,0) + f(0,1/2) + f(1/2,1) + f(1,1/2))$

$= L(f)$ above. Thus $L(f) = \frac{1}{3}f(1/2,1/2) + \frac{1}{6}(f(1/2,0) + f(0,1/2) + f(1/2,1) + f(1,1/2))$ does arise as the integral of a unique interpolant of the form $A_1x^2 + A_2y^2 + A_3x + A_4y + A_5$.

If $a = 1$, $b = 0$, $c = 0$, $d = 1$, however, then $\det(B) = 0$. Thus, in this case, the interpolant of the form $A_1x^2 + A_2y^2 + A_3x + A_4y + A_5$ is not unique.

It is also natural to ask whether the cubature rule

$T(f) = \frac{1}{4}(f(1/2,0)+f(0,1/2)+f(1/2,1)+f(1,1/2))$ equals the integral of a unique second degree interpolant to f of the form

$$p(x,y) = A_1xy + A_2x + A_3y + A_4$$

We wish to choose the A_j so that

$$p(P_k) = f(P_k),\ k = 1, ..., 4$$

The corresponding coefficient matrix is $A = \begin{pmatrix} 0 & a & 0 & 1 \\ 0 & 0 & b & 1 \\ c & 1 & c & 1 \\ d & d & 1 & 1 \end{pmatrix}$, and

$\det(A) = cab - ac - cdb - dab + dac + db$. Note that if $a = b = c = d = \frac{1}{2}$, then

$\det(A) = 0$. Also, if $a = 1$, $b = 0$, $c = 0$, $d = 1$(gives the vertices), then again $\det(A) = 0$. In either case, there is not a unique interpolant, $p(x,y)$, in general, and thus $T(f)$ does not arise as the integral of a unique interpolant of the form $A_1xy + A_2x + A_3y + A_4$.

4 Polygons in the Plane

We have already discussed cubature formulas over triangles as a special case of the n simplex, and with points on the boundary other than the vertices. We also discussed cubature formulas over the unit square, as a special case of the n cube, and with points on the boundary other than the vertices. If $n > 4$, and $T(f)$ is generated using the vertices, then the weighted combination $\lambda M(f) + (1-\lambda)T(f)$ is a poor CR. We examine the special case $n = 6$.

4.1 6 sided Regular Polygons

Let D be the regular hexagon with vertices

$(1+\sqrt{3},0), (1,1), (-1,1), (-1-\sqrt{3},0), (-1,-1), (1,-1)$. Then the Center of Mass of

$D = (0,0)$. Using

$\int_D f(x,y)dA = \int_{-1}^{1}\int_{-1}^{1} f(x,y)dydx + \int_{1}^{1+\sqrt{3}}\int_{(x-1-\sqrt{3})/\sqrt{3}}^{-(x-1-\sqrt{3})/\sqrt{3}} f(x,y)dydx + \int_{-1-\sqrt{3}}^{-1}\int_{-(x+1+\sqrt{3})/\sqrt{3}}^{(x+1+\sqrt{3})/\sqrt{3}} f(x,y)dydx$

we list some useful integrals:

Area$(D) = \int_D dA = 4+2\sqrt{3}$

$\int_D xdA = \int_D ydA = \int_D x^3dA = \int_D y^3dA = \int_D xydA = \int_D x^2ydA = \int_D xy^2dA = 0$

$\int_D x^2dA = \frac{16}{3}+3\sqrt{3}, \int_D y^2dA = \frac{4}{3}+\frac{1}{3}\sqrt{3}$. Let

$M(f) = (+2\sqrt{3})f(0,0)$,

$T(f) = (4+2\sqrt{3})\frac{1}{6}(f(1+\sqrt{3},0)+f(1,1)+f(-1,1)+f(-1-\sqrt{3},0)+f(-1,-1)+f(1,-1))$

$L_\lambda(f) = \lambda M(f)+(1-\lambda)T(f)$. It follows easily that L_λ is exact for $1, x, y$, for any λ.

$L_\lambda(x^2) = \left(4+2\sqrt{3}\right)\left(\frac{1}{6}-\frac{1}{6}\lambda\right)\left(\left(1+\sqrt{3}\right)^2+4+\left(-1-\sqrt{3}\right)^2\right)$, and

$L_\lambda(y^2) = 4\left(4+2\sqrt{3}\right)\left(\frac{1}{6}-\frac{1}{6}\lambda\right)$. To make L_λ exact for y^2, say, we need

$4\left(4+2\sqrt{3}\right)\left(\frac{1}{6}-\frac{1}{6}\lambda\right) = \frac{4}{3}+\frac{1}{3}\sqrt{3} \Rightarrow \lambda = -\frac{\frac{4}{3}+\sqrt{3}}{-\frac{8}{3}-\frac{4}{3}\sqrt{3}} = \frac{1}{4}\frac{4+3\sqrt{3}}{2+\sqrt{3}}$. But then

$L_\lambda(x^2) = \left(4+2\sqrt{3}\right)\left(\frac{1}{6}-\frac{1}{6}\lambda\right)\left(\left(1+\sqrt{3}\right)^2+4+\left(-1-\sqrt{3}\right)^2\right) =$

$\frac{7}{3}\sqrt{3}+5 \neq I(x^2) = \frac{16}{3}+3\sqrt{3}$. Hence L_λ is only exact for the class of linear polynomials.

5 Irregular Quadrilaterals

Let $D \subset R^2$ be the trapezoid with vertices $\{(0,0),(1,0),(0,1),(1,2)\}$(call them P_j).

$\int_D dA = \frac{3}{2}, \int_D xdA = \frac{5}{6}, \int_D ydA = \frac{7}{6} \Rightarrow$ Center of mass of D is $(5/9, 7/9)$.

Define the following linear functionals:

$$M(f) = \text{Area}(D)\ f(5/9,7/9) = \frac{3}{2}f(5/9,7/9)$$

$$T(f) = \text{Area}(D)(\frac{1}{4}\sum_{j=1}^{4} f(P_j)) = \frac{3}{8}(f(0,0)+f(1,0)+f(0,1)+f(1,2)$$

For fixed $\lambda,\ 0 \leq \lambda \leq 1$,

$$L_\lambda = \lambda M(f)+(1-\lambda)T(f)$$

To make L_λ exact for x, we have $L_\lambda(x) = \frac{1}{12}\lambda+\frac{3}{4} = \frac{5}{6} \Rightarrow \lambda = 1$. Then $L_\lambda(y) = \frac{1}{24}\lambda+\frac{9}{8} = \frac{7}{6}$. However, $L_\lambda(xy) = \frac{35}{54} \neq \int_0^1\int_0^{x+1} xydydx = \frac{17}{24}$. So

using the vertices of $\partial(D)$ for $T(f)$ only gives exactness for linear functions in general.

Remark 10 *It is interesting to see if there are **any** weights w_1, w_2, w_3, w_4, w_5 such that the CR $L(f) = w_1 f(5/9, 7/9) + w_2 f(0,0) + w_3 f(1,0) + w_4 f(0,1) + w_5 f(1,2)$ is exact for all polynomials of degree ≤ 2. Above we considered the special case where the weights would have the form $w_1 = \frac{3}{2}\ \lambda$, $w_j = \frac{3}{8}(1-\lambda)$, $j = 2,3,4,5$. Here is the resulting linear system: $w_1 + w_2 + w_3 + w_4 + w_5 = \frac{3}{2}$, $\frac{5}{9} w_1 + w_3 + w_5 = \frac{5}{6}$, $\frac{7}{9} w_1 + w_4 + 2w_5 = \frac{7}{6}$, $\frac{25}{81} w_1 + w_3 + w_5 = \frac{7}{12}$,*
$\frac{49}{81} w_1 + w_4 + 4w_5 = \frac{5}{4}$, $\frac{35}{81} w_1 + 2w_5 = \frac{17}{24}$

The augmented matrix is $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \frac{3}{2} \\ \frac{5}{9} & 0 & 1 & 0 & 1 & \frac{5}{6} \\ \frac{7}{9} & 0 & 0 & 1 & 2 & \frac{7}{6} \\ \frac{25}{81} & 0 & 1 & 0 & 1 & \frac{7}{12} \\ \frac{49}{81} & 0 & 0 & 1 & 4 & \frac{5}{4} \\ \frac{35}{81} & 0 & 0 & 0 & 2 & \frac{17}{24} \end{pmatrix}$, *and the RREF of A is*

$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. *Hence the system has no solution, and thus there are **no** weights w_1, w_2, w_3, w_4, w_5 such that the CR $L(f) = w_1 f(5/9, 7/9) + w_2 f(0,0) + w_3 f(1,0) + w_4 f(0,1) + w_5 f(1,2)$ is exact for all*

polynomials of degree ≤ 2. One can, however, choose weights so that $L(f)$ is exact for all polynomials of degree ≤ 2, except for xy, say. The corresponding augmented matrix is

$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \frac{3}{2} \\ \frac{5}{9} & 0 & 1 & 0 & 1 & \frac{5}{6} \\ \frac{7}{9} & 0 & 0 & 1 & 2 & \frac{7}{6} \\ \frac{25}{81} & 0 & 1 & 0 & 1 & \frac{7}{12} \\ \frac{49}{81} & 0 & 0 & 1 & 4 & \frac{5}{4} \end{pmatrix}$, *which has row echelon form:* $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{81}{80} \\ 0 & 1 & 0 & 0 & 0 & \frac{23}{240} \\ 0 & 0 & 1 & 0 & 0 & \frac{17}{120} \\ 0 & 0 & 0 & 1 & 0 & \frac{29}{240} \\ 0 & 0 & 0 & 0 & 1 & \frac{31}{240} \end{pmatrix}$

5.0.1 Other points on Boundary of Trapezoid

We will try using other points on $\partial(D)$ to generate $T(f)$—say $(a,0),(0,b),(1,c),(d,d+1)$, with $0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 2, 0 \le d \le 1$. Then

$$L_\lambda(f) = \lambda\frac{3}{2}f(5/9,7/9) + (1-\lambda)\frac{3}{8}(f(a,0)+f(1,c)+f(0,b)+f(d,d+1)$$

- $L_\lambda(x) = \frac{5}{6}\lambda + \frac{3}{8}(1-\lambda)(a+1+d)$
- $L_\lambda(y) = \frac{7}{6}\lambda + \frac{3}{8}(1-\lambda)(c+b+d+1)$
- $L_\lambda(xy) = \frac{35}{54}\lambda + \frac{3}{8}(1-\lambda)(c+d(d+1))$
- $L_\lambda(x^2) = \frac{25}{54}\lambda + \frac{3}{8}(1-\lambda)(a^2+1+d^2)$
- $L_\lambda(y^2) = \frac{49}{54}\lambda + \frac{3}{8}(1-\lambda)\left(c^2+b^2+(d+1)^2\right)$

We set each of the above expressions in λ, a, b, c, d equal to the corresponding integral of f over D to yield the system:
$\frac{5}{6}\lambda + \frac{3}{8}(1-\lambda)(a+1+d) = \frac{5}{6}, \frac{7}{6}\lambda + \frac{3}{8}(1-\lambda)(c+b+d+1) = \frac{7}{6}$
$\frac{35}{54}\lambda + \frac{3}{8}(1-\lambda)(c+d(d+1)) = \frac{17}{24}, \frac{25}{54}\lambda + \frac{3}{8}(1-\lambda)(a^2+1+d^2) = \frac{7}{12}$
$\frac{49}{54}\lambda + \frac{3}{8}(1-\lambda)\left(c^2+b^2+(d+1)^2\right) = \frac{5}{4}$.
We used Maple to find the following Grobner basis:

$$\{392\lambda-163, 9a+9d-11, 81b-99d+20, 180d-191+81c, -22671d+6583+18549d^2\}$$

The last equation has solutions $d = \frac{11}{18} \pm \frac{1}{458}\sqrt{3893} \approx .74734, .47488$

We shall use $d = \frac{11}{18} + \frac{1}{458}\sqrt{3893}$. Solving the other equations yields
$180d - 191 + 81c = 0 \Rightarrow c = 1 - \frac{10}{2061}\sqrt{3893} \approx .69726$
$81b - 99d + 20 = 0 \Rightarrow b = \frac{1}{2} + \frac{11}{4122}\sqrt{3893} \approx .6665$
$9a + 9d - 11 = 0 \Rightarrow a = \frac{11}{18} - \frac{1}{458}\sqrt{3893} \approx .47488$
$392\lambda - 163 = 0 \Rightarrow \lambda = \frac{163}{392} \approx .41582$. We summarize

CR5: Let $D \subset R^2$ be the trapezoid with vertices $\{(0,0),(1,0),(0,1),(1,2)\}$. Let $a = \frac{11}{18} - \frac{1}{458}\sqrt{3893} \approx .47488$, $b = \frac{1}{2} + \frac{11}{4122}\sqrt{3893} \approx .6665$, $c = 1 - \frac{10}{2061}\sqrt{3893} \approx .69726$, $d = \frac{11}{18} + \frac{1}{458}\sqrt{3893} \approx .74734$, and $\lambda = \frac{163}{392} \approx .41582$. Then $L(f) = \lambda\frac{3}{2}f(5/9,7/9) + (1-\lambda)\frac{3}{8}(f(a,0)+f(1,c)+f(0,b)+f(d,d+1))$ is exact for all polys. of degree ≤ 2.

Note that $L(x^3) = \frac{336001}{762048} \approx .44092$, while $I(x^3) = \frac{9}{20} = .45$. So L is not exact for cubics.

6 Unit Disc

Let D be the unit disc. The n roots of unity $z_k = e^{2\pi ik/n}, k = 1, ..., n$, can be used to generate $T(f) = \frac{\pi}{n}\sum_{k=1}^{n} f(\cos(\ 2\pi k/n, \sin(2\pi k/n))$. Since the center of mass of $D = (0,0)$ and area$(D) = \pi$, the analogy of our formulas for the triangle and the square is $M(f) = \pi f(0,0)$, $L_\lambda(f) = \lambda\pi f(0,0) + (1-\lambda)\frac{\pi}{n}\sum_{k=1}^{n} f(\cos(\ 2\pi k/n, \sin(2\pi k/n))$. It is not hard to show, however, that the only good choice is $n = 4$. This gives the formulas

$$L_\lambda(f) = \lambda\pi f(0,0) + (1-\lambda)\frac{\pi}{4}(f(1,0) + f(0,1) + f(-1,0) + f(0,-1))$$

We list the following integrals without proof:

- If m and n are even whole numbers, then

$$\int\int_D x^m y^n dA = \frac{\pi}{m+n+2}\left(\frac{(n-1)!(m-1)!}{2^{m+n-3}(\frac{n}{2}-1)!(\frac{m}{2}-1)!(\frac{m+n}{2})!}\right)$$

- If m is an even whole number and $n = 0$, then

$$\int\int_D x^m y^n dA = \frac{\pi}{m+2}\left(\frac{(m-1)!}{2^{m-2}(\frac{m}{2}-1)!(\frac{m}{2})!}\right)$$

- If n is an even whole number and $m = 0$, then

$$\int\int_D x^m y^n dA = \frac{\pi}{n+2}\left(\frac{(n-1)!}{2^{n-2}(\frac{n}{2}-1)!(\frac{n}{2})!}\right)$$

- If m and/or n is an odd whole number, then

$$\int\int_D x^m y^n dA = 0$$

It is then easy to prove that the choice $\lambda = \frac{1}{2}$ gives

CR6: $L(f) = \frac{\pi}{2}f(0,0) + \frac{\pi}{8}(f(1,0) + f(0,1) + f(-1,0) + f(0,-1))$

is exact for all polynomials of degree ≤ 3.

7 Summary of Cubature Rules

These are the CRs we derived as a generalization of Simpson's Rule for functions of one variable. If D_n is a polygonal region in R^n, let $P_0, ..., P_n$ denote the $n+1$ vertices of D_n and P_{n+1} = Center of Mass of D_n. The generalization is based on the weighted combination $L_\lambda = \lambda M(f) + (1-\lambda)T(f)$, where $M(f) = \text{Vol}(D_n)\ f(P_{n+1})$, $T(f) = Vol(D_n)(\frac{1}{m+1}\sum_{j=0}^{m} f(P_j))$.

Unless noted otherwise, all of the rules have the following property:

All of the weights are positive, all of the knots lie inside the region, and all but one of the knots lies on the boundary of the region. It is desirable to have as many knots as possible on $\partial(D_n)$ if one subdivides the region and compounds the CR.

7.1 n Simplex T_n

- Using the vertices, $L(f) = \frac{n+1}{(n+2)n!} f(P_{n+1}) + \frac{1}{(n+2)!}\sum_{j=0}^{n} f(P_j)$ is exact for all polynomials of degree ≤ 2.
- Using points other than the vertices: Letting $\{Q_k\}$ denote the center of mass of the faces of $T_n, L(f) = -(n-2)\frac{n+1}{n+2}\frac{1}{n!} f(1/(n+1), ..., 1/(n+1)) + \frac{n^2}{(n+2)!}\sum_{k=1}^{n+1} f(Q_k)$ is exact for all polynomials of degree ≤ 2. All but one weight is positive if $n > 2$.

7.2 Unit n Cube C_n

- General n: Let $P_0, ..., P_{m-1}$ denote the m vertices of $C_n, m = 2^n$. Then $L(f) = \frac{2}{3} f(1/2, ..., 1/2) + \frac{1}{3}\frac{1}{2^n}\sum_{j=0}^{2^n-1} f(P_j)$ is exact for all polynomials of degree ≤ 3.
- $n = 2 : L(f) = \frac{1}{3} f(1/2, 1/2) + \frac{1}{6}(f(1/2, 0) + f(0, 1/2) + f(1/2, 1) + f(1, 1/2))$ is exact for all polynomials of degree ≤ 3.

7.3 Unit Disc D

$L(f) = \frac{\pi}{2} f(0,0) + \frac{\pi}{8}(f(1,0) + f(0,1) + f(-1,0) + f(0,-1))$ is exact for all polynomials of degree ≤ 3.

7.4 A Trapezoid in the Plane

Let $D \subset R^2$ be the trapezoid with vertices $\{(0,0),(1,0),(0,1),(1,2)\}$.

Let $a = \frac{11}{18} - \frac{1}{458}\sqrt{3893} \approx .47488$, $b = \frac{1}{2} + \frac{11}{4122}\sqrt{3893} \approx .6665$,

$c = 1 - \frac{10}{2061}\sqrt{3893} \approx .69726$, $d = \frac{11}{18} + \frac{1}{458}\sqrt{3893} \approx .74734$, and $\lambda = \frac{163}{392} \approx .41582$. Then $L(f) = \lambda\frac{3}{2}f(5/9, 7/9) + (1-\lambda)\frac{3}{8}(f(a,0) + f(1,c) + f(0,b) + f(d, d+1))$ is exact for all polys. of degree ≤ 2.

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